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Elimination of Corners in the Mapping of a Closed Curve

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SUMMARY

A transformation, which maps the exterior or the interior of a simple closed curve with corners into the exterior or interior respectively of a simple smooth (corner-free) closed curve, is introduced. Symmetry properties are shown to be preserved by the transformation and a numerical procedure for applying the proposed transformation to an arbitrary curve is presented.

Smooth curves, resulting from the application of the corner-eliminating transformation to a square and to a six-cornered double ship section are also given.

1. Introduction

A polygon with n corners can be mapped into a straight line by means of the Schwarz–Christoffel transformation. The ogive, with two corners, can be mapped into a circle by means of a known transformation. In both cases corners are eliminated by introducing transformations with branch points of proper order at the corner points. For simple, closed curves with more than two corners, however, a corner-eliminating transformation is not available.

Several problems in ship hydrodynamics require the mapping of a transverse double ship section into a circle. But double ship sections commonly have six corner points: two each at the top and bottom, and two at the non-normal intersection of the hull with the free surface. Thus it is desirable to derive a preliminary transformation which maps the ship section into a smooth (corner-free), simple closed curve.

A transformation which accomplishes this objective will be presented. It will be shown that if the original curve has double symmetry (as does a double ship section), then the smooth curve is also doubly symmetric. The results of transforming a square and a ship section in this manner will be shown.

2. The Transformations

Consider a simple, closed curve G with corners of exterior angles α_j, $j=1, 2, \ldots, n$, at the noncoincident points represented by the complex numbers a_j in the complex z-plane. Let the origin lie in the interior of G. We shall show that the curve G is mapped one-to-one into a simple, smooth, closed curve Γ in the ζ-plane ($\zeta=\xi+i\eta$) by the transformation

$$\left.\begin{aligned} \zeta &= c + \sum_{j=1}^{n} a_j p_j \ln z + \int_{z_0}^{z} F(z)dz \\ F(z) &= \prod_{j=1}^{n} \left(1 - \frac{a_j}{z}\right)^{p_j}, \quad p_j = \frac{\pi}{\alpha_j} - 1 \end{aligned}\right\} \tag{1}$$

and that $\zeta(z)$ is regular in the exterior of G, which is mapped one-to-one into the exterior of Γ. Here z_0 is an arbitrary reference point on or exterior to G, and c an arbitrary constant.

It is clear that $\zeta(z)$ has algebraic branch points at $z=a_j$ of order π/α_j, $j=1, 2, \ldots, n$, since, in a small neighborhood of a_j, the integrand of (1) may be written asymptotically in the form

$$\left(1 - \frac{a_j}{z}\right)^{p_j} E(a_j)\,.$$

The transformation is regular at the point $z=\infty$ which maps into the point at infinity in the ζ-plane. This is seen by writing (1) in the asymptotic form

$$\zeta \simeq \sum_{j=1}^{n} a_j p_j \ln z + \int_{z_0}^{z_1} F(z)dz + \int_{z_1}^{z} \left(1 - \frac{1}{z} \sum_{j=1}^{n} a_j p_j\right) dz \simeq z + c$$

where c is a constant and z_1 is a fixed point such that $|z_1| \gg a_j$, $j=1, 2, \ldots, n$, and $|z| > |z_1|$. Since there are no other branch points, and no singularities in the exterior of G, it is clear that, if the curve G, considered as a cut, is not crossed, the mapping between the exteriors of G and Γ is one-to-one. This then implies that the closed curve G maps one-to-one into Γ. Furthermore, the order of a branch point, π/α_j, indicates that an exterior angle α_j at a_j of G becomes an angle of π at the corresponding point of Γ; i.e., the corner is eliminated. Thus the transformation (1) has the desired properties for an exterior mapping.

Similarly, it can be shown that the interior of G is mapped one-to-one into the interior of a simple, smooth, closed curve Γ', by the transformation

$$\zeta = \int_{0}^{z} \prod_{j=1}^{n} \left(1 - \frac{z}{a_j}\right)^{q_j} dz\,, \quad q_j = \frac{\pi}{\beta_j} - 1 \tag{2}$$

where $\beta_j = 2\pi - \alpha_j$ is the interior angle of G at a_j. This transformation is regular in the interior of G and maps G one-to-one into Γ'. The origin in the z-plane corresponds to the origin in the ζ-plane.

It is instructive to compare the transformations (1) or (2) with that of Schwarz–Christoffel polygons. The latter is of the form

$$z = \int^{\zeta} \prod_{j=1}^{n} (\zeta - \lambda_j)^{r_j} d\zeta\,, \quad r_j = 1 - \frac{\alpha_j}{\pi} \tag{3}$$

$$z(a_j) = \zeta(\lambda_j)\,, \qquad j = 1, 2, \ldots, n\,. \tag{3a}$$

Here (3) is essentially a differential equation for $\zeta(z)$, in which the n unknown constants λ_j are to be determined from the auxiliary conditions (3a). The solution of these equations for arbitrary polygons is a difficult problem. In contrast, the transformations (1) and (2) are direct and require only simple numerical quadrature; the result is, of course less rewarding. The smooth curves yielded by transformations (1) or (2) would only serve as a preliminary step in mapping the original curve into a circle.

3. Symmetry Properties

First let us consider the case that G is symmetrical about the x-axis. The integrand of (1) is then seen to involve polynomials in $1/z$ with real coefficients, since the a_j are either real or occur in pairs of complex conjugates, and for such a pair we have

$$\left(1 - \frac{a_j}{z}\right)\left(1 - \frac{\bar{a}_j}{z}\right) = \left(1 - \frac{a_j + \bar{a}_j}{z} + \frac{a_j \bar{a}_j}{z^2}\right). \tag{4}$$

When z is real the above product becomes $|1 - a_j/x|^2$, which is real and positive, as is also the factor $(1 - a_j/x)$ with a_j real, since x is exterior to G. For the p_j-th power of these positive quantities we may select that branch which is also real. Thus we see from (1) that $\zeta(z)$ is real when z is real; i.e., the real axis in the z-plane is mapped into the real axis in the ζ-plane. Hence, by the Schwarz reflection principle, conjugate points in the z-plane map into conjugate points in the ζ-plane. Consequently the curve Γ must also be symmetrical about the x-axis.

Next let us suppose that G is symmetrical about both the x- and y-axes, as is the case for double ship sections. The integrand of (1) is now seen to involve polynomials in $1/z^2$ with real coefficients since, if a_j is real, we have

$$\left(1 - \frac{a_j}{z}\right)\left(1 + \frac{a_j}{z}\right) = \left(1 - \frac{a_j^2}{z^2}\right).$$

If a_j is imaginary, (4) becomes $(1-|a_j|^2/z^2)$, and if a_j is complex, then there are also corners at $-a_j$ and $\pm\bar{a}_j$, which yield

$$\left(1-\frac{a_j}{z}\right)\left(1+\frac{a_j}{z}\right)\left(1-\frac{\bar{a}_j}{z}\right)\left(1+\frac{\bar{a}_j}{z}\right)=1-\frac{a_j^2+\bar{a}_j^2}{z^2}+\frac{a_j^2\bar{a}_j^2}{z^4}. \tag{5}$$

These polynomials in $1/z^2$ are real and positive when z is real, and hence selection of that branch of the p_j-power of these polynomials which is real when z is real demonstrates that the integrand of (1) is also real. This also demonstrates that $F(z)$ is an even function.

Since both a_j and $-a_j$ occur, then $\Sigma_{j=1}^{n} a_j p_j=0$. Hence another consequence of the double symmetry of G is that the second term of (1) vanishes. The transformation (1) then becomes

$$\zeta=c+\int_{z_0}^{z} F(z)dz \tag{6}$$

which satisfies the condition $\zeta(z_0)=c$. Let us choose c so that $\zeta(-z_0)=-c$. Applying this last condition in (6) yields

$$c=\tfrac{1}{2}\int_{-z_0}^{z_0} F(z)dz \tag{7}$$

and substituting this value into (6) gives

$$\zeta=\tfrac{1}{2}\int_{-z_0}^{z} F(z)dz+\int_{z_0}^{z} F(z)dz \tag{8}$$

Since $F(z)$ is an even function, we have

$$\zeta=\left\{\tfrac{1}{2}\int_{z_0}^{z}-\tfrac{1}{2}\int_{-z}^{z}-\tfrac{1}{2}\int_{z_0}^{-z}+\int_{z_0}^{-z}+\int_{-z}^{z}\right\}F(z)dz=\tfrac{1}{2}\int_{-z}^{z} F(z)dz \tag{9}$$

a result which shows that the apparent dependence of ζ on z_0 in (8) is illusory. Since $F(z)$ is real when z is real, then $\zeta(z)$ is also real; i.e., the real axis in the z-plane is mapped into the real axis in the ζ-plane. The Schwarz reflection principle then indicates that symmetry about the real axis is preserved by the transformation (8). This, together with the result in (9) that $\zeta(z)$ is an odd function, completes the proof that symmetry about both the real and imaginary axes is preserved.

Similarly one may show that the transformation (2) for the interior mapping also preserves double symmetry.

4. Numerical Evaluation

We wish to determine the smooth curve Γ corresponding to a given doubly-symmetric curve G. We shall suppose that G is given in parametric form by

$$z(\sigma)=x(\sigma)+iy(\sigma) \tag{10}$$

with $z(\sigma_j)=a_j$. Let $z=a$ and $z=ib$ denote the intersections of G with the positive x- and y-axes respectively, and $\sigma=0$ correspond to $z=a$.

In order to determine the constant c in (6), we take $z_0=a$ in (6) and then observe from (7), (8), and (9) that $\zeta(a)=c$. Hence c is real, since ζ is real when z is real. When $z=ib$ we obtain from (6)

$$\zeta(ib)=c+\int_{a}^{ib} F(z)dz$$

Since $\zeta(ib)$ is also imaginary, this yields

$$c=-\mathrm{Re}\int_{a}^{ib} F(z)dz \tag{11}$$

where Re denotes the real part.

Thus c can be obtained from the line integral over G in the first quadrant. Let us write (6) in the form

$$\zeta(\sigma) = c + \int_a^{z(\sigma)} F(z)\frac{dz}{d\sigma}\,d\sigma \tag{12}$$

and subdivide each interval $\sigma_j \leqq \sigma \leqq \sigma_{j+1}$ into small, uniform subintervals $\Delta\sigma_j$. This defines a discrete set of values of $z(\sigma)$ on G and a corresponding discrete set of values of $\zeta(\sigma)$ on Γ. Since $F(z)$ is composed of several factors, each of which is multi-valued, it is necessary to select the proper branch of each of these factors and to remain on the selected branches as σ is varied. Two cases may occur: the point $z=a$ is not a branch point, or the point $z=a$ is a branch point and $a=a_1$. For the former case, the initial argument θ of each one of the factors of $F(z)$ at $z=a$ may be selected in the interval $-\pi+\theta_0 \leqq \theta < \pi+\theta_0$, where θ_0 is an arbitrary constant. A convenient choice is $\theta_0=0$. For the latter case, the initial argument of the factor $(1-a_1/z)$ at the point $z=a$ is $\alpha_1/2$ and the arguments of the other factors are again selected in the interval $-\pi \leqq \theta < \pi$.

Once the proper branch for each factor has been selected, the argument of each factor varies continuously with increasing σ until the next branch point is encountered. In crossing a branch point at $z=a_j$, the argument of the factor $(1-a_j/z)$ jumps by α_j, but the other factors of $F(z)$ remain continuous. One can now proceed in this manner through the successive branches of G.

Another difficulty, encountered when the integral (2) is evaluated by a quadrature formula, is that the integrand becomes infinite when $\alpha_j > \pi$, as is the case unless the corner is a re-entrant one. These singularities can be removed by the procedure of subtracting from the integrand functions having the identical singularities, but which can be integrated in closed form, as will now be described.

Let us consider the mapping of a particular branch of G lying between two branch points at $z=a_k$ and $z=a_{k+1}$. Equation (6) may then be written as

$$\zeta = \zeta_k + \int_{a_k}^{z} F(z)dz\ ; \quad \zeta_k = \zeta(a_k)\,, \ \zeta_1 = c + \int_a^{a_1} F(z)dz\,. \tag{13}$$

Two singularities are encountered in $F(z)$ at $z=a_k$ and $z=a_{k+1}$ when α_k and α_{k+1} are both larger than π. These singularities may be eliminated when (13) is written as

$$\begin{aligned}\zeta = \zeta_k &+ \int_{a_k}^{z} \left[F(z) - G_k(z-a_k)^{p_k} - G_{k+1}(z-a_{k+1})^{p_{k+1}}\right]dz \\ &+ \frac{\alpha_k}{\pi} G_k(z-a_k)^{\pi/\alpha_k} + \frac{\alpha_{k+1}}{\pi} G_{k+1}\left[(z-a_{k+1})^{\pi/\alpha_{k+1}} - (a_k-a_{k+1})^{\pi/\alpha_{k+1}}\right]\end{aligned} \tag{14}$$

where

$$G_m = \prod_{\substack{j=1\\ j\neq m}}^{n} a_m^{-p_j}\left(1-\frac{a_j}{a_m}\right)^{p_j}, \qquad m=k,\,k+1\,. \tag{15}$$

Since the proper branch of each one of the factors $(1-a_j/z)$ in $F(z)$ has already been selected, the argument of $(1-a_j/a_k)$ is then also determined. In addition we have

$$\operatorname{Arg}(z-a_i) = \operatorname{Arg}\left(1-\frac{a_i}{z}\right) + \operatorname{Arg}(z)\,. \tag{16}$$

Since $\operatorname{Arg}(1-a_k/z)$ is known for each z on the segment (a_k, a_{k+1}), $\operatorname{Arg}(z-a_k)$ is uniquely determined by (16) if we select

$$-\pi \leqq \operatorname{Arg}(z) < \pi\,. \tag{17}$$

The integral in (14) may then be replaced by a quadrature formula and the computation is straightforward. The foregoing procedure will be illustrated in two numerical examples.

5. Example 1: Mapping of a square

For the exterior mapping of the square shown in Fig. 1, we have $\alpha_j = 3\pi/2$, $p_j = -\frac{1}{2}$, and

$$F(z) = \left(1 - \frac{1+\mathrm{i}}{z}\right)^{-\frac{1}{2}} \left(1 + \frac{1-\mathrm{i}}{z}\right)^{-\frac{1}{2}} \left(1 + \frac{1+\mathrm{i}}{z}\right)^{-\frac{1}{2}} \left(1 - \frac{1-\mathrm{i}}{z}\right)^{-\frac{1}{2}} . \tag{18}$$

At the point $A(1,0)$, the initial arguments of the four factors of $F(z)$ are $(-\pi/2, -\mathrm{tg}^{-1}\frac{1}{2}, \mathrm{tg}^{-1}\frac{1}{2}, \pi/2)$. These arguments determine the branch of each one of the factors of $F(z)$.

A simplification may be introduced in the present case by substituting $z = 1 + \mathrm{i}y$, $0 \leqq y \leqq 1$ in (18) for points along the line AB. Then (18) becomes

$$\left.\begin{aligned} F(z) &= \frac{(1+y^2)^{\frac{2}{3}}}{(1-y^2)^{\frac{1}{3}}(y^4+6y^2+25)^{\frac{1}{6}}}\, \mathrm{e}^{\mathrm{i}\theta(y)} \\ \theta(y) &= \tfrac{4}{3}\, \mathrm{tg}^{-1}\, y - \tfrac{1}{3}\, \mathrm{tg}^{-1} \frac{4y}{5-y^2} \end{aligned}\right\} \tag{19}$$

The arguments of the four factors of $F(z)$ in (18), as $y \to 1$ from below, are given by $(-3\pi/4, -\pi/4, 0, \pi/4)$. Hence the selected branch for $F(z)$ in (19) requires $\theta(0) = 0$ and $\theta(1) = \pi/4$.

The singularity at $a_1 = 1 + \mathrm{i}$ can be eliminated by writing

$$\int_0^{\mathrm{i}y} F(z)dz = \int_0^y \frac{G(y) - yG(1)}{(1-y^2)^{\frac{1}{3}}}\, dy + G(1) \int_0^y \frac{y}{(1-y^2)^{\frac{1}{3}}}\, dy$$

where

$$G(y) = \frac{\mathrm{i}(1+y^2)^{\frac{2}{3}}}{(y^4+6y^2+25)^{\frac{1}{6}}}\, \mathrm{e}^{\mathrm{i}\theta(y)} .$$

Then

$$G(1) = 2^{-\frac{1}{6}}\, \mathrm{e}^{3\pi\mathrm{i}/4} .$$

The real and imaginary parts of $\zeta - c$ for $0 \leqq y \leqq 1$ can now be computed for uniform increments of y by a quadrature formula.

The mapping of the remainder of the square in the first quadrant, BC, can be derived immediately from that for AB, since the mapping is symmetric about the 45-degree radial in the ζ-plane. This can be shown by first verifying that the line $x = y$ in the z-plane maps into the line $\xi = \eta$ in the ζ-plane, and then applying the Schwarz reflection principle. Because of this symmetry, c can be obtained from (11) in the form

$$c = -2\,\mathrm{Re} \int_0^1 \mathrm{i}F(\mathrm{i}y)dy . \tag{20}$$

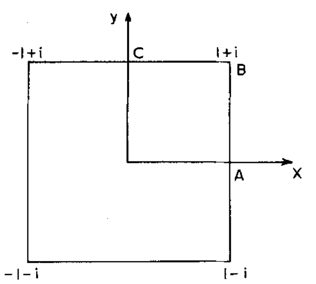

Figure 1. Notation for the square.

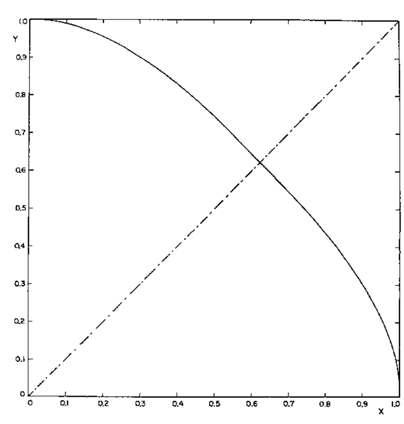

Figure 2. Transformation of the square.

The resulting smooth curve Γ, obtained by using a 50-point Simpson-rule quadrature formula, is shown in Fig. 2.

6. Example 2: Mapping of a ship section

A double ship section with a flat bottom and a nonnormal intersection with the free surface (Fig. 3) is defined by the polar-coordinate data given in Table 1, with $z=r\,e^{i\theta}$. Corner points occur at $a_1=a$ and $a_2=d+ib$ in the first quadrant. Let us consider the mapping of the first branch. From (14) we obtain

$$\begin{aligned}\zeta(\phi)=c+&\int_0^{\phi}\left[F(r\,e^{i\theta})-G_1(r\,e^{i\theta}-a)^{p_1}-G_2(r\,e^{i\theta}-a_2)^{p_2}\right]\left[\frac{dr}{d\theta}+ir\right]e^{i\theta}d\theta\\&+\frac{\alpha_1}{\pi}G_1(z-a)^{\pi/\alpha_1}+\frac{\alpha_2}{\pi}G_2\left[(z-a_2)^{\pi/\alpha_2}-(a-a_2)^{\pi/\alpha_2}\right]\qquad 0\leqq\phi\leqq\phi_2\end{aligned}\tag{21}$$

where $z=r(\phi)e^{i\phi}$ and

$$F(z)=\left(1-\frac{a}{z}\right)^{p_1}\left(1-\frac{a_2}{z}\right)^{p_2}\left(1+\frac{\bar{a}_2}{z}\right)^{p_2}\left(1+\frac{a}{z}\right)^{p_1}\left(1+\frac{a_2}{z}\right)^{p_2}\left(1-\frac{\bar{a}_2}{z}\right)^{p_2}\tag{22}$$

$$G_1=(a/2)^{-p_1}\left(1-\frac{a_2}{a}\right)^{p_2}\left(1+\frac{a_2}{a}\right)^{p_2}\left(1-\frac{\bar{a}_2}{a}\right)^{p_2}\left(1+\frac{\bar{a}_2}{a}\right)^{p_2}\tag{23}$$

$$G_2=(a_2/2)^{-p_2}\left(1-\frac{a}{a_2}\right)^{p_1}\left(1+\frac{a}{a_2}\right)^{p_1}\left(1-\frac{\bar{a}_2}{a_2}\right)^{p_2}\left(1+\frac{\bar{a}_2}{a_2}\right)^{p_2}.\tag{24}$$

TABLE 1

Polar representation of the ship section

	ϕ	r		ϕ	r		ϕ	r		ϕ	r
1	0.0000	0.3888	21	1.1223	0.3109	41	1.3875	0.5698	61	1.4711	0.8445
2	0.0379	0.3697	22	1.1534	0.3217	42	1.3927	0.5835	62	1.4748	0.8583
3	0.0798	0.3513	23	1.1806	0.3332	43	1.3976	0.5972	63	1.4786	0.8720
4	0.1264	0.3334	24	1.2051	0.3449	44	1.4024	0.6109	64	1.4825	0.8858
5	0.1778	0.3168	25	1.2261	0.3571	45	1.4071	0.6246	65	1.4869	0.8995
6	0.2346	0.3012	26	1.2452	0.3696	46	1.4116	0.6383	66	1.4920	0.9132
7	0.2963	0.2878	27	1.2622	0.3822	47	1.4160	0.6520	67	1.4977	0.9268
8	0.3614	0.2773	28	1.2768	0.3951	48	1.4204	0.6658	68	1.5044	0.9404
9	0.4294	0.2691	29	1.2901	0.4081	49	1.4246	0.6795	69	1.5122	0.9540
10	0.4994	0.2632	30	1.3021	0.4213	50	1.4288	0.6932	70	1.5156	0.9594
11	0.5700	0.2595	31	1.3132	0.4345	51	1.4329	0.7070	71	1.5193	0.9649
12	0.6410	0.2576	32	1.3232	0.4478	52	1.4370	0.7207	72	1.5233	0.9703
13	0.7098	0.2579	33	1.3324	0.4612	53	1.4410	0.7345	73	1.5277	0.9757
14	0.7756	0.2600	34	1.3410	0.4747	54	1.4449	0.7482	74	1.5325	0.9811
15	0.8392	0.2635	35	1.3491	0.4881	55	1.4488	0.7620	75	1.5382	0.9865
16	0.8984	0.2685	36	1.3565	0.5018	56	1.4526	0.7757	76	1.5451	0.9919
17	0.9537	0.2748	37	1.3635	0.5152	57	1.4564	0.7895	77	1.5536	0.9973
18	1.0028	0.2824	38	1.3701	0.5288	58	1.4601	0.8032	78	1.5596	1.0000
19	1.0475	0.2910	39	1.3762	0.5424	59	1.4638	0.8170			
20	1.0873	0.3006	40	1.3820	0.5561	60	1.4675	0.8308			

For the mapping of the flat part we write

$$\begin{aligned}\zeta(\phi)=c+\zeta(\phi_2)+&\int_{\phi_2}^{\pi/2}\left[F(r\,e^{i\theta})-G_2(r\,e^{i\theta}-a_2)^{p_2}\right]\left[\frac{dr}{d\theta}+ir\right]e^{i\theta}d\theta\\&+\frac{\alpha_2}{\pi}G_2(z-a_2)^{\pi/\alpha_2}\qquad\frac{\pi}{2}\geqq\phi\geqq\phi_2\,.\end{aligned}\tag{25}$$

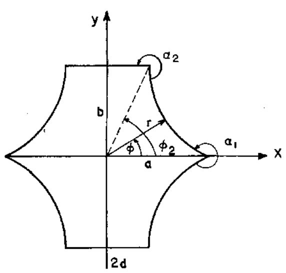

Figure 3. Notation for a double ship section.

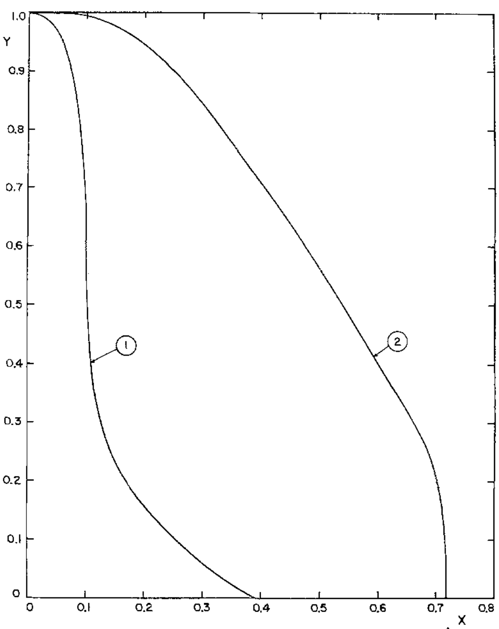

Figure 4. Comparison between the original (1) and transformed (2) ship section.

The initial arguments of the factors in each one of the six parentheses in (22), evaluated at $z=a$, are $(\alpha_1/2, -\beta, -\gamma, 0, \gamma, \beta)$ respectively, where $\beta=\text{tg}^{-1}[b/(a-d)]$ and $\gamma=\text{tg}^{-1}[b/(a+d)]$ (see Fig. 3 for notation). These arguments vary continuously until $\phi=\phi_2$. When passing through the branch point at $z=a_2$, the argument of the term in the second parenthesis in (22) is incremented by α_2 while the other arguments remain continuous when passing through this branch point.

Following our convention, we have taken $0<\text{Arg}(z)<\pi/2$ in the first quadrant, and hence

$$\text{Arg}(z-a_2)=\text{Arg}(1-a_2/z)+\text{Arg}(z)$$

uniquely determines $\text{Arg}(z-a_2)$. Once the proper branch for each one of the terms and factors in (21) through (24) has been selected, the integrals (21) and (25) may be evaluated by a quadrature formula.

The foregoing procedure was applied to the ship section defined by the data in Table 1 and shown in Fig. 4. The input data consist of 78 points (r, ϕ) in the interval $\phi=0$ to $\phi_2=89.3°$, which were interpolated using the Lagrange five-point interpolation formula to obtain 450 points in the same interval. A Lagrange five-point interpolation formula was also used to compute the discrete values of $dr/d\theta$. The quadratures were performed by the Simpson rule using 450 points for $\phi=0$ to $\phi_2=89.3°$ and 5 points from $\phi_2=89.3°$ to 90°. The resulting smooth curve is also shown in Fig. 4.

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